Multi-Domain Transmission Conditions for Domain Decomposition Methods Applied to Scattering Problems

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The choice of transmission conditions parameters has a significant impact on the convergence of domain decomposition methods, especially when applied to exterior propagation and scattering problems. In this work, we study numerically the convergence behavior of two domain decomposition algorithms within the framework of the Finite Element Method, namely the Restricted Additive Schwarz and Multiplicative Schwarz with upward and downward sweeps (or Double Sweep), with different transmission conditions for the electromagnetic scattering problem in 2d. In addition, we propose a new approach to obtain the optimized parameters based on the optimization of the spectral radius of the iteration matrix on the Fourier domain.

Index Terms—**Finite Element, Electromagnetic Scattering, Domain Decomposition, Schwarz Methods**

I. Introduction

THE NUMERICAL simulation of wave propagation and scattering problems is present in several fields of engineering and science (antenna design, medical imaging, geophysics..etc), but at high frequencies it still poses major computational challenges [1]. As the frequency grows the wavelength becomes small compared to the computational domain, and the number of degrees of freedom per wavelength to represent the oscillations of the solution and control the pollution effect becomes larger. On the one hand, direct solvers may require extensive memory and time requirements; on the other hand, iterative solvers based on Krylov subspace methods may present slow convergence or even diverge [1].

Domain Decomposition Methods (DDM) appear as an appealing alternative because of their suitability for working on parallel computers and guaranteed convergence as iterative solvers (with two sub-domains and specific transmission conditions) [2]. Optimized transmission conditions can be used to improve convergence [2], [3], but even with optimal transmission condition the number of iterations grows proportionally to the number of sub-domains [4].

The Transmission Conditions (TC) are applied at the interfaces between neighbor sub domains, and therefore the exchange of information is restricted locally. To further improve the convergence, the addition of a new component that allows global communication becomes necessary [2] - [5]. Unfortunately, the basic principle of the coarse grid fails when employed for propagating problems. Furthermore, the standard procedure for obtaining free TC parameters considers only two sub-domains and does not take into account that the contribution of the propagation modes remain significant at long distances [3].

In view of that, we aim to analyze the influence of the number of sub-domains on the rate of convergence of Schwarz decomposition method with and without a coarse grid and propose a new approach to obtain optimized TC that take into account the number of sub-domains and the propagating nature of the problem.

A. Model Problem

We consider the normal incidence plane wave scattering by an infinitely long (in z direction) perfect conducting object (see Fig. 1) and assume that the wave has TM^z polarization. The problem can be written as:

$$\left.\begin{array}{rcc} \Delta E_z^t + k^2 E_z^t = 0 & \textit{in} & \Omega \\ E_z^t = 0 & \textit{on} & \Gamma_d \\ \nabla E_z^t \cdot \hat{\mathbf{n}} - jkE_z^t = \nabla E_z^i \cdot \hat{\mathbf{n}} - jkE_z^i & \textit{on} & \Gamma_\infty \end{array}\right\} \quad (1)$$

where $E_z^t = E_z^i + E_z^s$ is the total electric field, $E_z^i = E_0 e^{-jkx}$ is the incident (plane wave field) and E_z^s is scattered field due to the presence of the scatters.

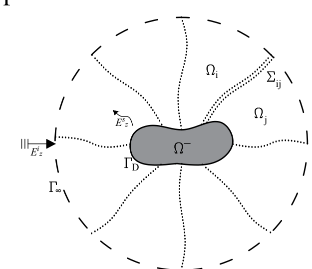

Fig. 1. An exterior scattering problem with circular artificial truncation boundary decomposed into multiple subdomains

II. Domain Decomposition - Optimized Schwarz

The original problem in (1) is decomposed into n smaller boundary value problems:

$$\left.\begin{array}{rcc} (\Delta + k^2)E_{zi} = 0 & \textit{in} & \Omega_i \\ \mathcal{B}_{ij}(E_{zi}) = \mathcal{B}_{ji}(E_{zj}) & \textit{in} & \Sigma_{ij} \end{array}\right\} \quad (2)$$

The boundary conditions on $\partial\Omega_i \cap \partial\Omega$ from the original problem are retained, although they are omitted in the description of subproblem in (2). We assume that the linear operators $\mathcal{B}$ are of the form:

$$\mathcal{B} = \partial_n + \mathcal{S}_j \quad (3)$$

The optimal choice of the transmission operator $\mathcal{S}_j$ in (3) is the Dirichlet-to-Neumann (DtN) map [2]. The DtN is a non-local operator, and therefore optimized Schwarz DDM make use of use a local approximation of this operator. This amounts to defining absorbing boundary conditions on the artificial interfaces. A perfectly matched layer (PML) can also be used for that purpose, as in [5].

The propagating modes are the main reason of the non-scalability with respect to the number of sub-domains, because the propagating modes can go much further than the evanescent modes. We will concentrate on overlapping DDM, because the overlap can take care of the higher spatial frequencies due to the exponential decay [3].

A. 0th Order Approximation

The most common TC is the known as Robin or 0th Order TC, and can be written as:

$$\mathcal{S}^{app} = \pm(p + qi), \quad p, q \in \mathbf{R}^{+} \tag{4}$$

We will compare three approaches to obtain the free parameters p and q in (4). The first TO_0 (Taylor Zeroth Order), is obtained by the Taylor expansion of the DtN symbol in the vicinity of $f = 0$ (where f is the frequency in the Fourier domain). As a result we have that $p = 0$ and $q = k$. The second is OO_0 (Zeroth Order optimized), which is based on the Fourier transform of the partial differential equation with two sub-domains, obtaining an explicit recurrence relation for the iteration. The free parameters are then obtained by optimizing the convergence factor [2]. With some simplifying assumptions closed formulas for p, q can be obtained [3].

In this paper we propose a different approach, MO_0 (Multiple Domain Optimized Order Zero TC). Instead of considering only two sub-domains after Fourier transform of the partial differential equation, we consider n layered sub-domains, which results in an iteration matrix Ψ, of size $2n \times 2n$. Then the free parameters are obtained by optimizing the spectral radius of Ψ. More details on the algorithm and numerical analysis of second-order transmission conditions will be provided in the full paper.

III. Numerical Results

We use a model problem with a known exact solution, a plane wave scattering by an infinitely long perfect conducting metallic cylinder with $r = 0.5m$. A first order absorbing boundary conditions (ABC) with a fictitious circular boundary Γ_∞ with radius $R = 1.5m$ is used to truncate the domain. The incident wave is a plane wave e^{jkx}. Two different frequencies were analyzed $k = 2\pi$ and $k = 10\pi$. In both cases the discretization density is $n_\lambda = \frac{\lambda}{h} = 25$, using linear triangular finite elements. DDM methods has been also applied to high order edge elements [6].

The computational domain Ω is decomposed into n overlapping sub-domains. Each Ω_i is obtained by extending a non-overlapping partition by one element mesh layer. The local finite element matrices are assembled, with the aid of the FEniCS library [7]. Each sub-system is solved only once, using a multifrontal LU factorization [8], and then in each iteration backsubstituitons steps are performed. Fig. 2 shows the cyclic domain partition for $k = 10\pi$ and $n = 32$. Emphasis will be given to the convergence of the methods, measured by the iteration count when a relative 10^{-6} decrease of the residual is reached. The relative error remained below 3.0% for all experiments.

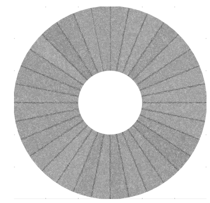

Fig. 2. Decomposition into 32 subdomains with minimal overlap.

Both Restricted Additive Schwarz (RAS) and an overlapping modified version of the Double Sweep algorithms are used as preconditioners for the GMRES method. Iteration counts for $k = 2\pi$ are presented in Table I. The MO_0 TC performed better than the others in spite of the algorithm used. The same conclusions can be drawn for $k = 10\pi$, see Table II.

TABLE I
$k = 2\pi$

	RAS			Double Sweep		
# Subdomains	OO_0	TO_0	MO_0	OO_0	TO_0	MO_0
8	37	33	31	5	5	5
16	54	49	48	16	15	9
32	95	90	88	20	19	16
64	127	123	119	26	26	20

TABLE II
$k = 10\pi$

	RAS			Double Sweep		
# Subdomains	OO_0	TO_0	MO_0	OO_0	TO_0	MO_0
8	54	47	47	15	12	11
16	75	67	66	36	31	31
32	190	142	142	95	88	77
64	331	268	264	155	90	84

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